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Handbook of Electrochemical Impedance Spectroscopy

CIRCUITS made of RESISTORS and CAPACITORS

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Chapter 1

Circuits made of one R and one C

1.1 Circuit (R+C)

The symbol ‘+’ denotes the serial association of electrical components.

\[ Z(\omega) = R + \frac{1}{C\omega} = R\frac{1+i\omega\tau}{i\omega\tau}, \quad \tau = RC \]

\[ \text{Re } Z(\omega) = R, \quad \text{Im } Z(\omega) = -\frac{R}{\tau\omega} \]

1.1.2 Reduced impedance

\[ Z^*(u) = \frac{Z}{R} = 1 + \frac{1}{iu} = \frac{1+iu}{iu}, \quad u = \tau\omega, \quad \text{Re } Z^*(u) = 1, \quad \text{Im } Z^*(u) = -\frac{1}{u} \]

(Fig. 1.2)

1.2 Circuit (R/C)

The symbol ‘/’ denotes the parallel association of electrical components.
CHAPTER 1. CIRCUITS MADE OF ONE R AND ONE C

Figure 1.2: Nyquist diagram of the reduced impedance and admittance \((Y^* = RY)\) for the (R+C) circuit (Fig. 1.1, Eq. (1.1)). The arrows always indicate the increasing frequency direction.

Figure 1.3: Circuit (R/C).

1.2.1 Impedance

\[
Z(\omega) = \frac{R}{1 + i \omega \tau}, \quad \tau = RC
\]

\[
\text{Re } Z(\omega) = \frac{R}{1 + \tau^2 \omega^2}, \quad \text{Im } Z(\omega) = -\frac{R \omega \tau}{1 + \tau^2 \omega^2}
\]

1.2.2 Reduced impedance

\[
Z^*(u) = \frac{Z(u)}{R} = \frac{1}{1 + i u}, \quad u = \tau \omega, \quad \text{Re } Z^*(u) = \frac{1}{1 + u^2}, \quad \text{Im } Z^*(u) = -\frac{u}{1 + u^2}
\]

(Fig. 1.4)

Figure 1.4: Nyquist diagram of the reduced impedance and admittance \((Y^* = RY)\) for the (R/C) circuit (Fig. 1.3, Eq. (1.2)).
Chapter 2

Circuits made of two Rs and one C

2.1 Circuit \((R_2 + \frac{R_1}{C_1})\)

\[
\begin{align*}
    Z(\omega) &= R_2 + \frac{1}{i\omega C_1 + \frac{1}{R_1}} \\
    Z(\omega) &= \frac{(R_1 + R_2) (1 + i\omega \tau_2)}{1 + i\omega \tau_1}, \quad \tau_1 = R_1 C_1, \quad \tau_2 = \frac{C_1 R_1 R_2}{R_1 + R_2}
\end{align*}
\]

2.1.1 Impedance

\[
Z^*(u) = \frac{Z(u)}{R_1 + R_2} = \frac{1 + T iu}{1 + iu}
\]

\(u = \tau_1 \omega, \quad T = \frac{\tau_2}{\tau_1} = R_2/(R_1 + R_2) < 1\)

\[
\begin{align*}
    \text{Re } Z^*(u) &= \frac{1 + Tu^2}{1 + u^2}, \quad \text{Im } Z^*(u) = \frac{(T - 1) u}{1 + u^2} \\
    \lim_{u \to 0} \text{Re } Z^*(u) &= 1, \quad \lim_{u \to \infty} \text{Re } Z^*(u) = T
\end{align*}
\]

(Fig. 2.2)
2.2 Circuit \(((R_1+C_1)/R_2)\)

\[
\begin{align*}
\text{Figure 2.3: Circuit } ((R_1+C_1)/R_2). & \\
2.2.1 \text{ Impedance} & \\
Z(\omega) &= \frac{R_2 (1 + i \omega \tau_2)}{1 + i \omega \tau_1}, \quad \tau_1 = C_1 (R_1 + R_2) \ , \ \tau_2 = R_1 C_1 \\
\text{Re } Z(\omega) &= \frac{R_2 (1 + \omega^2 \tau_1 \tau_2)}{1 + \omega^2 \tau_1^2}, \quad \text{Im } Z(\omega) = \frac{\omega R_2 (-\tau_1 + \tau_2)}{1 + \omega^2 \tau_1^2}. \\
2.2.2 \text{ Reduced impedance} & \\
Z^*(u) &= \frac{Z(u)}{R_2} = \frac{1 + T i u}{1 + i u} \quad (2.2) \\
u = \tau_1 \omega, \ T = \tau_2/\tau_1 = R_1/(R_1 + R_2) < 1 \quad \text{cf. Eq. (2.1) and Fig. 2.2.} \\
2.3 \text{ Transformation formulae between } (R+(R/C)) \text{ and } ((R+C)/R) \text{ circuits} \\
2.3.1 \text{ Transformation formulae } (R+(R/C)) \rightarrow ((R+C)/R) & \\
R_{22} &= R_{11} + R_{21}, \ R_{12} = R_{11} \frac{R_{11}^2}{R_{21}}, \ C_{12} = \frac{C_{11} R_{21}^2}{(R_{11} + R_{21})^2}. 
\end{align*}
\]
2.3. TRANSFORMATION FORMULAE BETWEEN \((R+(R/C))\) AND \(((R+C)/R)\) CIRCUITS

Figure 2.4: The \((R+(R/C))\) and \(((R+C)/R)\) circuits are non-distinguishable [3, 2, 1, 6].

2.3.2 Transformation formulae \(((R+C)/R) \rightarrow (R+(R/C))\)

\[
C_{11} = \frac{C_{12}(R_{12} + R_{22})^2}{R_{22}^2}, \quad R_{11} = \frac{R_{12}R_{22}}{R_{12} + R_{22}}, \quad R_{21} = \frac{R_{22}^2}{R_{12} + R_{22}}
\]
CHAPTER 2. CIRCUITS MADE OF TWO RS AND ONE C
Chapter 3

Circuits made of one R and two Cs

3.1 Circuit \(((R_1/C_1)+C_2)\)

![Circuit diagram](image)

Figure 3.1: Circuit \(((R_1/C_1)+C_2)\).

3.1.1 Impedance

\[
Z(\omega) = \frac{1}{i\omega C_1 + \frac{1}{R_1}} + \frac{1}{i\omega C_2} = \frac{1 + i\omega (C_1 + C_2)}{i\omega C_2 (1 + i\omega C_1 R_1)}
\]

\[
Z(\omega) = \frac{1 + i\omega \tau_2}{i\omega C_2 (1 + i\omega \tau_1)}, \quad \tau_1 = R_1 C_1, \quad \tau_2 = (C_1 + C_2) R_1, \quad \tau_1 < \tau_2
\]

\[
\text{Re} \ Z(\omega) = -\frac{\tau_1 - \tau_2}{C_2 (1 + \omega^2 \tau_1^2)}, \quad \text{Im} \ Z(\omega) = -\frac{1 + \omega^2 \tau_1 \tau_2}{\omega C_2 (1 + \omega^2 \tau_1^2)}
\]

\[
\lim_{\omega \to 0} \text{Re} \ Z(\omega) = R_1
\]

3.1.2 Reduced impedance

\[
Z^*(u) = \frac{Z(u)}{R_1} = \frac{1 + T i u}{T - 1 i u (1 + i u)}\quad (3.1)
\]

\[
u = \omega \tau_1, \quad T = \frac{\tau_2}{\tau_1} = 1 + \frac{C_2}{C_1} > 1
\]
**CHAPTER 3. CIRCUITS MADE OF ONE R AND TWO CS**

\[
\begin{align*}
\text{Re } Z^*(u) &= \frac{1}{1+u^2}, \quad \text{Im } Z^*(u) = -\frac{1+u^2 T}{(T-1)u(1+u^2)} \\
\lim_{u \to 0} \text{Re } Z^*(u) &= 1
\end{align*}
\]

(Fig. 3.2)

Figure 3.2: Nyquist diagram of the reduced impedance and admittance \((Y^* = R Y)\) for the \(((R_1/C_1)+C_2)\) circuit (Fig. 3.1, Eq. (3.1)) plotted for \(T = 4, 9, 90\). The line thickness increases with increasing \(T\). Horizontal tangent is observed for \(T \geq 9\) \((C_2/C_1 \geq 8)\) [4]. Dots: reduced characteristic angular frequency \(u_{c1} = 1\); circles: reduced characteristic angular frequency \(u_{c2} = 1/T\).

### 3.2 Circuit \(((R_1+C_2)/C_1)\)

![Circuit Diagram](image)

Figure 3.3: Circuit \(((R_1+C_2)/C_1)\).

#### 3.2.1 Impedance

\[
Z(\omega) = \frac{1}{\omega C_1 + \frac{1}{R_1 + \frac{1}{1 + i\omega C_2}}} = \frac{1 + i\omega C_2 R_1}{i\omega (C_1 + C_2) \left(1 + \frac{i\omega C_1 C_2 R_1}{C_1 + C_2}\right)}
\]

\[
Z(\omega) = \frac{(1 + i\omega \tau_2)}{i\omega (C_1 + C_2) (1 + i\omega \tau_1)}, \quad \tau_1 = \frac{C_1 C_2 R_1}{C_1 + C_2}, \quad \tau_2 = C_2 R_1
\]  \(\text{(3.2)}\)
3.3 Transformation Formulae Between \(((R/C)/C)\) and \(((R+C)/C)\) Circuits

3.3.1 Transformation Formulae \(((R/C)/C)\) → \(((R+C)/C)\)

\[
\begin{align*}
C_{22} &= \frac{C_{11}C_{21}}{C_{11} + C_{21}}, \\
R_{12} &= \frac{(C_{11} + C_{21})^2 R_{11}}{C_{21}^2}, \\
C_{12} &= \frac{C_{21}^2}{C_{11} + C_{21}}
\end{align*}
\]

Figure 3.4: Nyquist diagram of the reduced impedance and admittance \((Y^* = R_1 Y)\) for the \(((R_1 + C_2)/C_1)\) circuit (Fig. 3.3, Eq. (3.3)) plotted for \(T = 4, 9, 90\). The line thickness increases with increasing \(T\). Horizontal tangent is observed for \(T \geq 9\). \((C_2/C_1 \geq 8)\). Dots: reduced characteristic angular frequency \(u_{c1} = 1\); circles: reduced characteristic angular frequency: \(u_{c2} = 1/T\).
3.3.2 Transformation formulae \(((R+C)/C) \rightarrow ((R/C)/C)\)

\[ C_{11} = C_{22} + \frac{C_{22}^2}{C_{12}}, \quad R_{11} = \frac{C_{12}^2 R_{12}}{(C_{12} + C_{22})^2}, \quad C_{21} = C_{12} + C_{22} \]
Chapter 4

Circuits made of two Rs and two Cs

4.1 Circuit \(((R_1/C_1)+(R_2/C_2))\)

\[\text{Figure 4.1: Circuit } ((R_1/C_1)+(R_2/C_2)).\]

4.1.1 Impedance

\[Z(\omega) = \frac{1}{i\omega C_1 + \frac{1}{R_1}} + \frac{1}{i\omega C_2 + \frac{1}{R_2}} = \frac{(R_1 + R_2) (1 + i\omega \tau_3)}{(1 + i\omega \tau_1) (1 + i\omega \tau_2)}\]

\[\tau_1 = R_1 C_1, \tau_2 = R_2 C_2, \tau_3 = \frac{(C_1 + C_2) R_1 R_2}{R_1 + R_2} = \frac{\tau_1 \tau_2 + \tau_1 R_1}{R_1 + R_2}\]

\[\text{Re } Z(\omega) = \frac{(R_1 + R_2) (1 + \omega^2 (-\tau_1 \tau_2 + (\tau_1 + \tau_2) \tau_3))}{(1 + \omega^2 \tau_1^2) (1 + \omega^2 \tau_2^2)}\]

\[\text{Im } Z(\omega) = -\frac{\omega (R_1 + R_2) (\tau_1 + \tau_2 + (-1 + \omega^2 \tau_1 \tau_2) \tau_3)}{(1 + \omega^2 \tau_1^2) (1 + \omega^2 \tau_2^2)}\]

4.1.2 Reduced impedance

\[Z^*(u) = Z(u) \frac{R_1 + R_2}{R_1 + R_2} = \frac{1 + \rho + (T + \rho) i u}{(1 + \rho) (1 + i u) (1 + i u T)}\]

\[u = R_2 C_2 \omega, \rho = R_1/R_2, T = R_1 C_1 / (R_2 C_2) = \gamma \rho, \gamma = C_1/C_2\]

(Figs. 4.2-4.5)
CHAPTER 4. CIRCUITS MADE OF TWO RS AND TWO CS

Figure 4.2: Case diagram for the \(((R_1/C_1)+(R_2/C_2))\) circuit plotted using the \(\log \rho \text{ vs. } \log T\) representation.

Figure 4.3: Array of impedance diagrams plotted for the \(((R_1/C_1)+(R_2/C_2))\) circuit, depending on \(\rho\) and \(T\) values, and enlargement of the high frequency part of the diagram calculated for \(T = 10^{-2}\) and \(\rho = 10^{-2}\).

Figure 4.4: Case diagram for the \(((R_1/C_1)+(R_2/C_2))\) circuit, using the \(\log \rho \text{ vs. } \log \gamma\) representation.
4.1. CIRCUIT \((R_1/C_1)+(R_2/C_2)\)

Figure 4.5: Array of impedance diagrams plotted for the \((R_1/C_1)+(R_2/C_2)\) circuit, depending on \(\rho\) and \(\gamma\) values and enlargement of the high frequency part of the diagram calculated for \(\gamma = 10^{-2}\) and \(\rho = 10^{-2}\).
CHAPTER 4. CIRCUITS MADE OF TWO RS AND TWO CS

4.2 Circuit \(\left( \frac{R_1 + (R_2 / C_2)}{C_1} \right)\)

![Circuit Diagram]

Figure 4.6: Circuit \(\left( \frac{R_1 + (R_2 / C_2)}{C_1} \right)\).

4.2.1 Impedance

\[
Z(\omega) = \frac{1}{i \omega C_1 + \frac{1}{R_1 + \frac{R_2}{1 + i \omega C_2 R_2}}} - \frac{(R_1 + R_2) \left( 1 + \frac{i \omega C_2 R_1 R_2}{R_1 + R_2} \right)}{1 + i \omega \left( C_2 R_2 + C_1 (R_1 + R_2) \right) + (i \omega)^2 C_1 C_2 R_1 R_2}
\]

\[
Z(\omega) = \frac{(R_1 + R_2) \left( 1 + \frac{i \omega \tau_3}{R_1 + R_2} \right)}{(1 + i \omega \tau_1) (1 + i \omega \tau_2)}, \quad \tau_3 = \frac{C_2 R_1 R_2}{R_1 + R_2}
\]

The poles of the impedance of a circuit made of Rs and Cs are always real [5].

\[
\tau_1 = \frac{C_2 R_2 + C_1 (R_1 + R_2) - \sqrt{-4 C_1 C_2 R_1 R_2 + (C_2 R_2 + C_1 (R_1 + R_2))^2}}{2}
\]

\[
\tau_2 = \frac{C_2 R_2 + C_1 (R_1 + R_2) + \sqrt{-4 C_1 C_2 R_1 R_2 + (C_2 R_2 + C_1 (R_1 + R_2))^2}}{2}
\]

(4.1)

(4.2)

4.2.2 Reduced impedance

\[
Z^*(u) = \frac{Z(u)}{R_1 + R_2} = \frac{1 + \rho (1 + i u)}{1 + \rho (1 + i u T + \rho (1 + i u) (1 + i u T))}
\]

\[
u = R_2 C_2 \omega, \quad \rho = R_1 / R_2, \quad T = R_1 C_1 / (R_2 C_2) = \gamma \rho, \quad \gamma = C_1 / C_2
\]

(Figs. 4.7-4.10)
4.2. CIRCUIT \((R_1 + (R_2/C_2))/C_1\)

Figure 4.7: Case diagram for the \(((R_1 + (R_2/C_2))/C_1)\) circuit plotted using the \(\log \rho\) vs. \(\log T\) representation.

Figure 4.8: Array of impedance diagrams plotted for the \(((R_1 + (R_2/C_2))/C_1)\) circuit, depending on \(\rho\) and \(T\) values, and enlargement of the high frequency part of the diagram calculated for \(T = 10^{-3}\) and \(\rho = 10^{-3}\).

Figure 4.9: Case diagram for the \(((R_1 + (R_2/C_2))/C_1)\) circuit plotted using the \(\log \rho\) vs. \(\log \gamma\) representation.
Figure 4.10: Array of impedance diagrams plotted for the \((R_1+(R_2/C_2))/C_1\) circuit, depending on \(\rho\) and \(\gamma\) values, and enlargement of the high frequency part of the diagram calculated for \(\gamma = 10^{-3}\) and \(\rho = 10^{-3}\).
4.3 Circuit \((\frac{C_1+(R_2/C_2))}{R_1})\)

![Circuit Diagram]

Figure 4.11: Circuit \((\frac{C_1+(R_2/C_2))}{R_1})\).

4.3.1 Impedance

\[
Z(\omega) = \frac{1}{\frac{R_1}{1 + \frac{1}{R_1} \frac{i\omega R_2}{C_1} + \frac{i\omega R_2}{C_2}}}
\]

\[
Z(\omega) = \frac{R_1 (1 + i\omega (C_1 + C_2) R_2)}{1 + i\omega (C_2 R_2 + C_1 (R_1 + R_2)) + (i\omega)^2 C_1 C_2 R_1 R_2}
\]

\[
Z(\omega) = \frac{R_1 (1 + i\omega \tau_3)}{(1 + i\omega \tau_1)(1 + i\omega \tau_2)}, \quad \tau_3 = (C_1 + C_2) R_2, \quad \tau_1 : \text{Eq. (4.1)}, \quad \tau_2 : \text{Eq. (4.2)}
\]

4.3.2 Reduced impedance

\[
Z^*(u) = Z(u)/R_1 = \frac{\rho + i u (\rho + T)}{iuT + \rho (1 + i u) (1 + i u T)}
\]

\[
u = R_2 C_2 \omega, \quad \rho = R_1/R_2, \quad T = R_1 C_1/(R_2 C_2) = \gamma \rho, \quad \gamma = C_1/C_2
\]

(Figs. 4.12 and 4.13)
CHAPTER 4. CIRCUITS MADE OF TWO RS AND TWO CS

Figure 4.12: Impedance diagrams array and case diagram for the \((C_1 + (R_2/C_2))/R_1\) circuit using the \(\log \rho \text{ vs. } \log T\) representation.

Figure 4.13: Impedance diagrams array and case diagram for the \((C_1 + (R_2/C_2))/R_1\) circuit using the \(\log \rho \text{ vs. } \log \gamma\) representation.
4.4 Circuit (((C₂+R₂)/R₁)/C₁)

![Circuit Diagram]

Figure 4.14: Circuit (((C₂+R₂)/R₁)/C₁).

4.4.1 Impedance

\[ Z(\omega) = \frac{1}{i \omega C₁ + \frac{R₁}{1} + \frac{1}{i \omega C₂ + R₂}} \]

\[ Z(\omega) = \frac{R₁ (1 + i \omega C₂ R₂)}{1 + i \omega (C₁ R₁ + C₂ (R₁ + R₂)) + (i \omega)^2 C₁ C₂ R₁ R₂} \]

\[ Z(\omega) = \frac{R₁ (1 + i \omega \tau₃)}{(1 + i \omega \tau₁)(1 + i \omega \tau₂)} \]

\[ \tau₃ = R₂ C₂; \ \tau₁ : \text{Eq. (4.1)}, \ \tau₂ : \text{Eq. (4.2)} \ (\text{exchanging subscripts 1 and 2}) \]

4.4.2 Reduced impedance

\[ Z^*(u) = Z(u)/R₁ = \frac{1 + i u}{1 + i u (1 + \rho + \tau (1 + i u))} \]

\[ u = R₂ C₂ \omega, \ \rho = R₁/R₂, \ T = R₁ C₁ / (R₂ C₂) = \gamma \rho, \ \gamma = C₁/C₂ \]

(Figs. 4.15 and 4.16)
Figure 4.15: Impedance diagrams array and case diagram for the \( ((C_2+R_2)/R_1)/C_1 \) circuit using the \( \log \rho \) vs. \( \log T \) representation.

Figure 4.16: Impedance diagrams array and case diagram for the \( ((C_2+R_2)/R_1)/C_1 \) circuit using the \( \log \rho \) vs. \( \log \gamma \) representation.
4.5 Transformation formulae for the four circuits made of two Rs and two Cs

\[ C_{11} = \frac{1}{2C_{22}R_{22}} \sqrt{C_{22}^2 R_{22}^2 + 2C_{12}C_{22}R_{22} (-R_{12} + R_{22}) + C_{12}^2 (R_{12} + R_{22})^2} \times \left( C_{22} R_{22} - C_{12} (R_{12} + R_{22}) + \sqrt{C_{22}^2 R_{22}^2 + 2C_{12}C_{22}R_{22} (-R_{12} + R_{22}) + C_{12}^2 (R_{12} + R_{22})^2} \right) \]

\[ C_{21} = \frac{1}{2C_{22}R_{22}} \sqrt{C_{22}^2 R_{22}^2 + 2C_{12}C_{22}R_{22} (-R_{12} + R_{22}) + C_{12}^2 (R_{12} + R_{22})^2} \times \left( -C_{22} R_{22} + C_{12} (R_{12} + R_{22}) + \sqrt{C_{22}^2 R_{22}^2 + 2C_{12}C_{22}R_{22} (-R_{12} + R_{22}) + C_{12}^2 (R_{12} + R_{22})^2} \right) \]
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\[ R_{21} = \left( C_{22} \left( R_{12} - R_{22} \right) R_{22} - C_{12} \left( R_{12} + R_{22} \right)^2 + \right) \]

\[
(R_{12} + R_{22}) \sqrt{C_{22}^2 R_{22}^2 + 2 C_{12} C_{22} R_{22} \left( -R_{12} + R_{22} \right) + C_{12}^2 \left( R_{12} + R_{22} \right)^2} / \\
\left( 2 \sqrt{C_{22}^2 R_{22}^2 + 2 C_{12} C_{22} R_{22} \left( -R_{12} + R_{22} \right) + C_{12}^2 \left( R_{12} + R_{22} \right)^2} \right)
\]

\[ R_{11} = \frac{1}{2} \left( R_{12} + R_{22} + \frac{C_{22} R_{22} \left( -R_{12} + R_{22} \right) + C_{12} \left( R_{12} + R_{22} \right)^2}{\sqrt{C_{22}^2 R_{22}^2 + 2 C_{12} C_{22} R_{22} \left( -R_{12} + R_{22} \right) + C_{12}^2 \left( R_{12} + R_{22} \right)^2}} \right) \]

4.5.2 Transformation formulae circuit 1 \( \rightarrow \) circuit 2

\[ C_{12} = \frac{C_{11} C_{21}}{C_{11} + C_{21}}, C_{22} = \frac{(C_{11}^2 R_{11} + C_{21}^2 R_{21})^2}{(C_{11} + C_{21}) (C_{11} R_{11} - C_{21} R_{21})^2} \]

\[ R_{12} = \frac{(C_{11} + C_{21})^2 R_{11} R_{21}}{C_{11}^2 R_{11} + C_{21}^2 R_{21}}, R_{22} = \frac{(C_{11} R_{11} - C_{21} R_{21})^2}{C_{11}^2 R_{11} + C_{21}^2 R_{21}} \]

4.5.3 Transformation formulae circuit 3 \( \rightarrow \) circuit 1

\[ C_{11} = \frac{1}{2 C_{13}^2 R_{13}^2} \left( C_{13}^2 \left( C_{13} + C_{23} \right) R_{13}^2 + 2 C_{13} \left( C_{13}^2 - C_{23}^2 \right) R_{13} R_{23} + \right) \]

\[
(C_{13} + C_{23})^3 R_{23}^2 - \left( C_{13} \left( C_{13} - C_{23} \right) R_{13} + (C_{13} + C_{23})^2 R_{23} \right) \times \sqrt{C_{23}^2 R_{23}^2 + 2 C_{13} C_{23} R_{23} \left( -R_{13} + R_{23} \right) + C_{13}^2 \left( R_{13} + R_{23} \right)^2} \]

\[ C_{21} = \frac{1}{2 C_{13}^2 R_{13}^2} \left( C_{13}^2 \left( C_{13} + C_{23} \right) R_{13}^2 + 2 C_{13} \left( C_{13}^2 - C_{23}^2 \right) R_{13} R_{23} + \right) \]

\[
(C_{13} + C_{23})^3 R_{23}^2 + \left( C_{13} \left( C_{13} - C_{23} \right) R_{13} + (C_{13} + C_{23})^2 R_{23} \right) \times \sqrt{C_{23}^2 R_{23}^2 + 2 C_{13} C_{23} R_{23} \left( -R_{13} + R_{23} \right) + C_{13}^2 \left( R_{13} + R_{23} \right)^2} \]

\[ R_{21} = R_{13} \left( C_{13} R_{13} - (C_{13} + C_{23}) R_{23} + \right) \]

\[
\sqrt{C_{23}^2 R_{23}^2 + 2 C_{13} C_{23} R_{23} \left( -R_{13} + R_{23} \right) + C_{13}^2 \left( R_{13} + R_{23} \right)^2} / \\
\left( 2 \sqrt{C_{23}^2 R_{23}^2 + 2 C_{13} C_{23} R_{23} \left( -R_{13} + R_{23} \right) + C_{13}^2 \left( R_{13} + R_{23} \right)^2} \right) \]

\[ R_{11} = \frac{R_{13}}{2} \left( 1 + \frac{-C_{13} R_{13} + (C_{13} + C_{23}) R_{23}}{\sqrt{C_{23}^2 R_{23}^2 + 2 C_{13} C_{23} R_{23} \left( -R_{13} + R_{23} \right) + C_{13}^2 \left( R_{13} + R_{23} \right)^2}} \right) \]
4.5. TRANSFORMATION FORMULAE FOR RRCC CIRCUITS

4.5.4 Transformation formulae circuit 1 → circuit 3

\[ R_{13} = R_{11} + R_{21}, \quad C_{23} = \frac{C_{11} C_{21} (C_{11} R_{11}^2 + C_{21} R_{21}^2)}{(C_{11} R_{11} - C_{21} R_{21})^2} \]

\[ C_{13} = \frac{C_{11} R_{11}^2 + C_{21} R_{21}^2}{(R_{11} + R_{21})^2}, \quad R_{23} = \frac{R_{11} R_{21} (R_{11} + R_{21}) (C_{11} R_{11} - C_{21} R_{21})^2}{(C_{11} R_{11}^2 + C_{21} R_{21}^2)^2} \]

4.5.5 Transformation formulae circuit 4 → circuit 1

\[ C_{11} = \frac{1}{2 C_{24} R_{14}} \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2} \]
\times (C_{14} R_{14} - C_{24} (R_{14} + R_{24}) + \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2})

\[ C_{21} = \frac{1}{2 C_{24} R_{14}} \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2} \]
\times (-C_{14} R_{14} + C_{24} (R_{14} + R_{24}) + \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2})

\[ R_{21} = (R_{14} ((C_{14} + C_{24}) R_{14} - C_{24} R_{24} + \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2})) / \]
\left(2 \sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2}\right)

\[ R_{11} = \frac{R_{14}}{2} \left(1 + \frac{- (C_{14} + C_{24}) R_{14} + C_{24} R_{24}}{\sqrt{C_{14} R_{14}^2 + 2 C_{14} C_{24} R_{14} (R_{14} - R_{24}) + C_{24}^2 (R_{14} + R_{24})^2}}\right) \]

4.5.6 Transformation formulae circuit 1 → circuit 4

\[ C_{14} = \frac{C_{11} C_{21}}{C_{11} + C_{21}}, \quad R_{14} = R_{11} + R_{21} \]

\[ R_{24} = \frac{(C_{11} + C_{21})^2 R_{11} R_{21} (R_{11} + R_{21})}{(C_{11} R_{11} - C_{21} R_{21})^2}, \quad C_{24} = \frac{(C_{11} R_{11} - C_{21} R_{21})^2}{(C_{11} + C_{21}) (R_{11} + R_{21})^2} \]

4.5.7 Transformation formulae circuit 3 → circuit 2

\[ C_{12} = \frac{C_{13} C_{23}}{C_{13} + C_{23}}, \quad C_{22} = \frac{C_{13}^2 R_{13} + (C_{13} + C_{23})^2 R_{23}}{C_{13}^2 (C_{13} + C_{23}) R_{13}^2} \]

\[ R_{12} = \frac{(C_{13} + C_{23})^2 R_{13} R_{23}}{C_{13}^2 R_{13} + (C_{13} + C_{23})^2 R_{23}}, \quad R_{22} = \frac{C_{13}^2 R_{13}^2}{C_{13}^2 R_{13} + (C_{13} + C_{23})^2 R_{23}} \]
4.5.8 Transformation formulae circuit 2 → circuit 3

\[ R_{13} = R_{12} + R_{22}, C_{23} = \frac{C_{12} \left( C_{22} R_{22}^2 + C_{12} (R_{12} + R_{22})^2 \right)}{C_{22} R_{22}^2} \]

\[ C_{13} = C_{12} + \frac{C_{22} R_{22}^2}{(R_{12} + R_{22})^2}, R_{23} = \frac{C_{22}^2 R_{12} R_{22}^3 (R_{12} + R_{22})}{\left( C_{22} R_{22}^2 + C_{12} (R_{12} + R_{22})^2 \right)^2} \]

4.5.9 Transformation formulae circuit 4 → circuit 2

\[ C_{12} = C_{14}, C_{22} = \frac{C_{24} (R_{14} + R_{24})^2}{R_{14}^2}, R_{12} = \frac{R_{14} R_{24}}{R_{14} + R_{24}}, R_{22} = \frac{R_{14}^2}{R_{14} + R_{24}} \]

4.5.10 Transformation formulae circuit 2 → circuit 4

\[ C_{14} = C_{12}, R_{14} = R_{12} + R_{22}, R_{24} = R_{13} + \frac{R_{12}^2}{R_{22}}, C_{24} = \frac{C_{22} R_{22}^2}{(R_{12} + R_{22})^2} \]

4.5.11 Transformation formulae circuit 4 → circuit 3

\[ R_{13} = R_{14}, C_{23} = C_{14} + \frac{C_{14}^2}{C_{24}}, C_{13} = C_{14} + C_{24}, R_{23} = \frac{C_{24} R_{24}}{(C_{14} + C_{24})^2} \]

4.5.12 Transformation formulae circuit 3 → circuit 4

\[ C_{14} = \frac{C_{13} C_{23}}{C_{13} + C_{23}}, R_{24} = \frac{(C_{13} + C_{23})^2 R_{23}}{C_{13}^2}, R_{14} = R_{13}, C_{24} = \frac{C_{13}^2}{C_{13} + C_{23}} \]
Bibliography


